Entanglement between Collective Operators in the Linear Harmonic Chain

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We investigate entanglement between collective operators of two blocks of oscillators in an infinite linear harmonic chain. These operators are defined as averages over local operators (individual oscillators) in the blocks. On the one hand, this approach of "physical blocks" meets realistic experimental conditions, where measurement apparatuses do not interact with single oscillators but rather with a whole bunch of them, i.e., where in contrast to usually studied "mathematical blocks" not every possible measurement is allowed. On the other, this formalism naturally allows the generalization to blocks which may consist of several non-contiguous regions. We quantify entanglement between the collective operators by a measure based on the Peres-Horodecki criterion and show how it can be extracted and transferred to two qubits. Entanglement between two blocks is found even in the case where none of the oscillators from one block is entangled with an oscillator from the other, showing genuine bipartite entanglement between collective operators. Allowing the blocks to consist of a periodic sequence of subblocks, we verify that entanglement scales at most with the total boundary region. We also apply the approach of collective operators to scalar quantum field theory.

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I. INTRODUCTION

Quantum entanglement is a physical phenomenon in which the quantum states of two or more systems can only be described with reference to each other, even though the individual systems may be spatially separated. This leads to correlations between observables of the systems that cannot be understood on the basis of classical (local realistic) theories [1]. Its importance today exceeds the realm of the foundations of quantum physics and entanglement has become an important physical resource, like energy, that allows performing communication and computation tasks with efficiency which is not achievable classically [2].

In the near future we will certainly see more and more experiments on entanglement of increasing complexity. Moving to higher entangled systems or entangling more systems with each other, will eventually push the realm of quantum physics well into the macroscopic world. It will be therefore important to investigate under which conditions entanglement within or between "macroscopic" objects, each consisting of a sample containing a large number of the constituents, can arise.

Recently, it was shown that macroscopic entanglement can arise "naturally" between constituents of various complex physical systems. Examples of such systems are chains of interacting spin systems [2, 3], harmonic oscillators [4, 5] and quantum fields [6]. Entanglement can have an effect on the macroscopic properties of these systems [7, 8, 9] and can be in principle extractable from them for quantum information processing [6, 10, 11, 12].

With the aim of better understanding macroscopical entanglement we will investigate entanglement between *collective operators* in this paper. A simple and natural system is the ground state of a linear chain of harmonic oscillators furnished with harmonic nearest-neighbor interaction. The mathematical entanglement properties of this system were extensively investigated in [4, 5, 13, 14]. Entanglement was computed in the form of logarithmic negativity for general bisections of the chain and for contiguous blocks of oscillators that do not comprise the whole chain. It was shown that the log-negativity typically decreases exponentially with the separation of the groups and that the larger the groups, the larger the maximal separation for which the log-negativity is non-zero [4]. It also was proven that an area law holds for harmonic lattice systems, stating that the amount of entanglement between two complementary regions scales with their boundary [15].

In a real experimental situation, however, we are typically not able to determine the complete mathematical amount of entanglement (as measured, e.g., by log-negativity) which is non-zero even if two blocks share only one arbitrarily weak entangled pair of oscillators. Our measurement apparatuses normally cannot resolve single oscillators, but rather interact with a whole bunch of them in one way, potentially even in *non-contiguous regions*, thus measuring certain *global properties*. Here we will study entanglement between "physical blocks" of harmonic oscillators — existing only if there is entanglement between the *collective operators* defined on the entire blocks — as a function of their size, relative distance and the coupling strength. Our aim is to quantify (experimentally accessible) entanglement between global properties of two groups of harmonic oscillators. Surpris-

ingly, we will see that such collective entanglement can be demonstrated even in the case where none of the oscillators from one block is entangled with an oscillator from the other block (i.e., it cannot be understood as a cumulative effect of entanglement between pairs of oscillators), which is in agreement with [4]. This shows the existence of bipartite entanglement between collective operators.

Because of the area law [15] the amount of entanglement is relatively small in the first instance. We suggest a way to overcome this problem by allowing the collective blocks to consist of a *periodic sequence of subblocks*. Then the total boundary region between them is increased and we verify that indeed a larger amount of entanglement is found for periodic blocks, where the entanglement scales at most with the *total* boundary region. We give an analytical approximation of this amount of entanglement and motivate how it can in principle be extracted from the chain [6, 10, 11, 12].

Methodologically, we will quantify the entanglement between collective operators of two blocks of harmonic oscillators by using a measure for continuous variable systems based on the Peres-Horodecki criterion [16, 17, 18, 19]. The collective operators will be defined as sums over local operators for all single oscillators belonging to the block. The infinite harmonic chain is assumed to be in the ground state and since the blocks do not comprise the whole chain, they are in a mixed state.

II. LINEAR HARMONIC CHAIN

We investigate a linear harmonic chain, where each of the N oscillators is situated in a harmonic potential with frequency ω and each oscillator is coupled with its neighbors by a harmonic potential with the coupling frequency Ω. The oscillators have mass m and their positions and momenta are denoted as $\overline{q}_i$ and $\overline{p}_i$, respectively. Assuming periodic boundary conditions ($\overline{q}_{N+1} \equiv \overline{q}_1$), the Hamiltonian thus reads [20]

$$H = \sum_{j=1}^{N} \left(\frac{\overline{p}_j^2}{2\,m} + \frac{m\,\omega^2\,\overline{q}_j^2}{2} + \frac{m\,\Omega^2\,(\overline{q}_j - \overline{q}_{j-1})^2}{2} \right). \quad (1)$$

We canonically go to dimensionless variables: $q_j \equiv C\,\overline{q}_j$ and $p_j \equiv \overline{p}_j / C$, where $C \equiv \sqrt{m\omega(1+2\,\Omega^2/\omega^2)^{1/2}}$ [14]. By this means the Hamiltonian becomes

$$H = \frac{E_0}{2} \sum_{j=1}^{N} (p_j^2 + q_j^2 - \alpha\, q_j\, q_{j+1})\,, \quad (2)$$

with the abbreviations $\alpha \equiv 2\,\Omega^2/(2\,\Omega^2+\omega^2)$ and $E_0 \equiv \sqrt{2\,\Omega^2+\omega^2}$. The coupling constant is restricted to values $0 < \alpha < 1$, where $\alpha \to 0$ in the weak coupling limit ($\Omega/\omega \to 0$) and $\alpha \to 1$ in the strong coupling limit ($\Omega/\omega \to \infty$).

In the language of second quantization the positions and momenta are converted into operators ($q_j \to \hat{q}_j$, $p_j \to \hat{p}_j$) and are expanded into modes of their annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$, respectively:

$$\hat{q}_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\,\nu(\theta_k)}} \left[\hat{a}(\theta_k)\, \mathrm{e}^{\mathrm{i}\,\theta_k\, j} + \text{H.c.}\right], \quad (3)$$

$$\hat{p}_j = \frac{-\mathrm{i}}{\sqrt{N}} \sum_{k=0}^{N-1} \sqrt{\frac{\nu(\theta_k)}{2}} \left[\hat{a}(\theta_k)\, \mathrm{e}^{\mathrm{i}\,\theta_k\, j} - \text{H.c.}\right]. \quad (4)$$

Here $\theta_k \equiv 2\,\pi\,k/N$ (with $k = 0, 1, ..., N-1$) is the dimensionless pseudo-momentum and $\nu(\theta_k) \equiv \sqrt{1-\alpha\cos\theta_k}$ is the dispersion relation. The annihilation and creation operators fulfil the well known commutation relation $\left[\hat{a}(\theta_k), \hat{a}^\dagger(\theta_{k'})\right] = \delta_{kk'}$, since $[\hat{q}_i, \hat{p}_j] = \mathrm{i}\,\delta_{ij}$ has to be guaranteed. The ground state (vacuum), denoted as $|0\rangle$, is defined by $\hat{a}(\theta_k)\,|0\rangle = 0$ holding for all θ_k. The two-point vacuum correlation functions

$$g_{|i-j|} \equiv \langle 0|\, \hat{q}_i\, \hat{q}_j\, |0\rangle \equiv \langle\, \hat{q}_i\, \hat{q}_j\, \rangle\,, \quad (5)$$

$$h_{|i-j|} \equiv \langle 0|\, \hat{p}_i\, \hat{p}_j\, |0\rangle \equiv \langle\, \hat{p}_i\, \hat{p}_j\, \rangle\,, \quad (6)$$

are given by $g_l = (2\,N)^{-1} \sum_{k=0}^{N-1} \nu^{-1}(\theta_k) \cos(l\,\theta_k)$ and $h_l = (2\,N)^{-1} \sum_{k=0}^{N-1} \nu(\theta_k) \cos(l\,\theta_k)$, where $l \equiv |i-j|$. In the limit of an infinite chain ($N \to \infty$) — which we will study below — and for $l < N/2$ they can be expressed in terms of the hypergeometric function ${}_2F_1$ [14]: $g_l = (z^l/2\,\mu)\binom{l-1/2}{l}\,{}_2F_1(1/2, l+1/2, l+1, z^2)$, $h_l = (\mu\,z^l/2)\binom{l-3/2}{l}\,{}_2F_1(-1/2, l-1/2, l+1, z^2)$, where $z \equiv (1-\sqrt{1-\alpha^2})/\alpha$ and $\mu \equiv 1/\sqrt{1+z^2}$.

III. DEFINING COLLECTIVE OPERATORS

In the following, we are interested in entanglement between two "physical blocks" of oscillators, where the blocks are represented by a specific form of *collective operators* which are normalized sums of individual operators. By means of such a formalism we seek to fulfil experimental conditions and constraints, since *finite experimental resolution implies naturally the measurement of, e.g., the average momentum of a bunch of oscillators rather than the momentum of only one.* On the other hand, this formalism can easily take account of blocks that *consist of non-contiguous regions*, leading to interesting results which will be shown below. We want to point out that this convention of the term *block* is not the same as it is normally used in the previous literature. In contrast to the latter, for which one allows any possible measurement, our simulation of realizable experiments already lacks some information due to the averaging.

Let us now consider two non-overlapping blocks of oscillators, A and B, within the closed harmonic chain in its ground state, where each block contains n oscillators. The blocks are separated by $d \geq 0$ oscillators (Fig. 1). We assume $n, d \ll N$ and $N \to \infty$ for the numerical calculations of the two-point correlation functions.

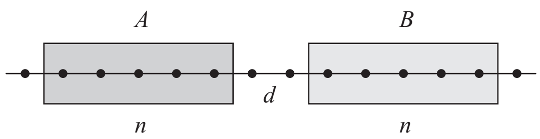

FIG. 1: Two blocks of a harmonic chain A and B. Each block consists of n oscillators and the blocks are separated by d oscillators.

By a Fourier transform we map the n oscillators of each block onto n ("orthogonal") *frequency-dependent* collective operators

$$\hat{Q}_A^{(k)} \equiv \frac{1}{\sqrt{n}} \sum_{j\in A} \hat{q}_j \, \mathrm{e}^{\frac{2\pi \mathrm{i}\, j\, k}{n}}, \tag{7}$$

$$\hat{P}_A^{(k)} \equiv \frac{1}{\sqrt{n}} \sum_{j\in A} \hat{p}_j \, \mathrm{e}^{-\frac{2\pi \mathrm{i}\, j\, k}{n}}, \tag{8}$$

with the frequencies $k = 0, ..., n-1$, and analogously for block B. The commutator of the collective position and momentum operators is

$$[\hat{Q}_A^{(k)}, \hat{P}_A^{(k')}] = \mathrm{i}\, \delta_{kk'} \,. \tag{9}$$

This means that collective operators for different frequencies $k \neq k'$ commute. For different blocks the commutator vanishes: $[\hat{Q}_A^{(k)}, \hat{P}_B^{(k')}] = 0$.

If the individual positions and momenta of all oscillators are written into a vector

$$\hat{\mathbf{x}} \equiv (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, ..., \hat{q}_N, \hat{p}_N)^{\mathrm{T}}, \tag{10}$$

then there holds the commutation relation

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\, \Omega_{ij} \tag{11}$$

with $\mathbf{\Omega}$ the n-fold direct sum of 2×2 symplectic matrices:

$$\mathbf{\Omega} \equiv \bigoplus_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12}$$

A matrix $\mathbf{S}$ transforms $\hat{\mathbf{x}}$ into a vector of collective (and uninvolved individual) oscillators:

$$\hat{\mathbf{X}} \equiv \mathbf{S}\, \hat{\mathbf{x}} = (\{\hat{Q}_A^{(k)}, \hat{P}_A^{(k)}\}_k, \{\hat{Q}_B^{(k)}, \hat{P}_B^{(k)}\}_k, \{\hat{q}_j, \hat{p}_j\}_j)^{\mathrm{T}}. \tag{13}$$

Here $\{\hat{Q}_A^{(k)}, \hat{P}_A^{(k)}\}_k = (\hat{Q}_A^{(0)}, \hat{P}_A^{(0)}, ..., \hat{Q}_A^{(n-1)}, \hat{P}_A^{(n-1)})$ denotes all collective oscillators of block A and analogously for block B, whereas $\{\hat{q}_j, \hat{p}_j\}_j$ denotes the $2\,(N-2\,n)$ position and momentum entries of those $N - 2\,n$ oscillators which are not part of one of the two blocks. The matrix $\mathbf{S}$ corresponds to a Gaussian operation [21]. It has determinant $\det \mathbf{S} = 1$ and preserves the symplectic structure

$$\mathbf{\Omega} = \mathbf{S}^{\mathrm{T}}\, \mathbf{\Omega}\, \mathbf{S}\,, \tag{14}$$

and hence

$$[\hat{X}_i, \hat{X}_j] = \mathrm{i}\, \Omega_{ij} \tag{15}$$

for all i, j, in particular verifying (9). This means that the Gaussianness of the ground state of the harmonic chain (i.e., the fact that the state is completely characterized by its first and second moments, see below) was preserved by the (Fourier) transformation to the frequency-dependent collective operators.

IV. QUANTIFYING ENTANGLEMENT BETWEEN COLLECTIVE OPERATORS

In reality, we are typically not capable of single particle resolution measurements and only of measuring the collective operators with one frequency, namely $k = 0$. Note that in general the correlations of higher-frequency collective operators, e.g., $\langle (\hat{Q}_A^{(k)})^2 \rangle$ or $\langle \hat{Q}_A^{(k)} \hat{Q}_B^{(k)} \rangle$ with $k \neq 0$, are not real numbers. Therefore, as a natural choice, we denote as the collective operators

$$\hat{Q}_A \equiv \hat{Q}_A^{(0)} = \frac{1}{\sqrt{n}} \sum_{j\in A} \hat{q}_j \,, \tag{16}$$

$$\hat{P}_A \equiv \hat{P}_A^{(0)} = \frac{1}{\sqrt{n}} \sum_{j\in A} \hat{p}_j \,, \tag{17}$$

and analogously for block B. It seems to be a very natural situation that the experimenter only has access to these collective properties and we are interested in the amount of (physical) entanglement one can extract from the system if only the collective observables $\hat{Q}_{A,B}$ and $\hat{P}_{A,B}$ are measured.

Reference [18] derives a separability criterion which is based on the Peres-Horodecki criterion [16, 17] and the fact that — in the continuous variables case — the partial transposition allows a geometric interpretation as mirror reflection in phase space. Following largely the notation in the original paper, we introduce the vector

$$\hat{\xi} \equiv (\hat{Q}_A, \hat{P}_A, \hat{Q}_B, \hat{P}_B) \tag{18}$$

of collective operators. The commutation relations have the compact form $[\hat{\xi}_\alpha, \hat{\xi}_\beta] = \mathrm{i}\, K_{\alpha\beta}$ with $\mathbf{K} \equiv \bigoplus_{j=1}^{2} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The separability criterion bases on the covariance matrix (of first and second moments)

$$V_{\alpha\beta} \equiv \frac{1}{2} \langle \Delta\hat{\xi}_\alpha \Delta\hat{\xi}_\beta + \Delta\hat{\xi}_\beta \Delta\hat{\xi}_\alpha \rangle \,, \tag{19}$$

where $\Delta\hat{\xi}_\alpha \equiv \hat{\xi}_\alpha - \langle \hat{\xi}_\alpha \rangle$ with $\langle \hat{\xi}_\alpha \rangle = 0$ in our case (state around the origin of phase space).

The covariance matrix $\mathbf{V}$ is real (which would not be the case for higher-frequency collective operators) and symmetric: $\langle \hat{Q}_A \hat{Q}_B \rangle = \langle \hat{Q}_B \hat{Q}_A \rangle$ and $\langle \hat{P}_A \hat{P}_B \rangle = \langle \hat{P}_B \hat{P}_A \rangle$, coming from the fact that the two-point correlation functions (5) and (6) only depend on the

absolute value of the position index difference. On the other hand, using (3) and (4), we verify that $\langle \hat{q}_i \, \hat{p}_j \rangle = \mathrm{i} \, (2\,N)^{-1} \sum_{k=0}^{N-1} \exp[\mathrm{i}\, \theta_k (i-j)]$ and $\langle \hat{p}_j \, \hat{q}_i \rangle = -\mathrm{i} \, (2\,N)^{-1} \sum_{k=0}^{N-1} \exp[\mathrm{i}\, \theta_k (j-i)]$. For $i \neq j$ both summations vanish ($\theta_k \equiv 2\,\pi\,k/N$ and i, j integer) and for $i = j$ they are the same but with opposite sign. Thus, in all cases $\langle \hat{q}_i \, \hat{p}_j \rangle = -\langle \hat{p}_j \, \hat{q}_i \rangle$. These symmetries also hold for the collective operators and hence we obtain

$$\mathbf{V} = \begin{pmatrix} G & 0 & G_{AB} & 0 \\ 0 & H & 0 & H_{AB} \\ G_{AB} & 0 & G & 0 \\ 0 & H_{AB} & 0 & H \end{pmatrix}. \tag{20}$$

The matrix elements are

$$G \equiv \langle \hat{Q}_A^2 \rangle = \langle \hat{Q}_B^2 \rangle = \frac{1}{n} \sum_{j \in A} \sum_{i \in A} g_{|j-i|} \,, \tag{21}$$

$$H \equiv \langle \hat{P}_A^2 \rangle = \langle \hat{P}_B^2 \rangle = \frac{1}{n} \sum_{j \in A} \sum_{i \in A} h_{|j-i|} \,, \tag{22}$$

$$G_{AB} \equiv \langle \hat{Q}_A \hat{Q}_B \rangle = \frac{1}{n} \sum_{j \in A} \sum_{i \in B} g_{|j-i|} \,, \tag{23}$$

$$H_{AB} \equiv \langle \hat{P}_A \hat{P}_B \rangle = \frac{1}{n} \sum_{j \in A} \sum_{i \in B} h_{|j-i|} \,. \tag{24}$$

To quantify entanglement between two collective blocks we use the degree of entanglement ε, given by the absolute sum of the negative eigenvalues of the partially transposed density operator: $\varepsilon \equiv \mathrm{Tr}|\rho^{\mathrm{T}_B}| - 1$, i.e., by measuring how much the mirror reflected state fails to be positive definite. This measure (negativity) is based on the Peres-Horodecki criterion [16, 17] and was shown to be an entanglement monotone [22, 23]. For covariance matrices of the form (20) it reads [19]

$$\varepsilon = \max\left(0, \frac{(\delta_1 \delta_2)_0}{\delta_1 \delta_2} - 1\right), \tag{25}$$

where $\delta_1 \equiv G - |G_{AB}|$ and $\delta_2 \equiv H - |H_{AB}|$. In general, the numerator is defined by the square of the Heisenberg uncertainty relation

$$(\delta_1 \delta_2)_0 \equiv \left(\frac{1}{2} \, |\langle [\hat{Q}_{A,B}, \hat{P}_{A,B}] \rangle|\right)^2 , \tag{26}$$

with $(\delta_1 \delta_2)_0 = 1/4$ due to (9). We note that ε is a *degree* of entanglement (in the sense of necessity and sufficiency) only for Gaussian states which are completely characterized by their first and second moments, as for example the ground state of the harmonic chain we are studying. However, we left out the higher-frequency collective operators (and all the oscillators which are not part of the blocks) and therefore, the entanglement ε has to be understood as the Gaussian part of the amount of entanglement which exists between (and can be extracted from) the two blocks when only the collective properties $\hat{Q}_{A,B}$ and $\hat{P}_{A,B}$, as defined in (16) and (17), are accessible.

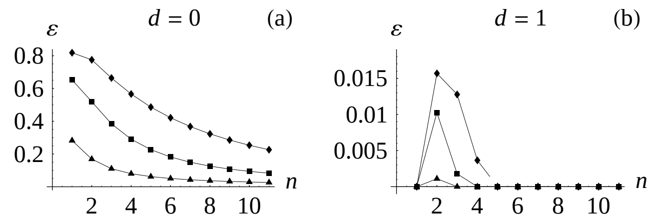

FIG. 2: Degree of collective entanglement ε for two blocks of oscillators as a function of their size n. (a) The blocks are neighboring ($d = 0$) and entanglement exists for all n and coupling strengths α. Plotted are $\alpha = 0.99$ (diamonds), $\alpha = 0.9$ (squares) and $\alpha = 0.5$ (triangles). (b) The same for two blocks which are separated by one oscillator ($d = 1$). The two blocks are unentangled for $n = 1$ but can be entangled, if one increases the block size ($n > 1$), although non of the individual pairs between the blocks is entangled.

There also exists an entanglement witness in form of a separability criterion based on variances, where $\Delta \equiv \langle (\hat{Q}_A - \hat{Q}_B)^2 \rangle + \langle (\hat{P}_A + \hat{P}_B)^2 \rangle = 2\,(G - G_{AB} + H + H_{AB}) < 2$ is a sufficient condition for the state to be entangled [24]. We note that the above negativity measure (25) is "stronger" than this witness in the whole parameter space (α, n). In particular, there are cases where $\varepsilon > 0$ although $\Delta \geq 2$. This is in agreement with the finding that the variance criterion is weaker than a generalized negativity criterion [25].

We further note that the amount of entanglement (25) is invariant under a change of potential redefinitions of the collective operators, e.g., $\hat{Q}_A \equiv \sum_{j \in A} \hat{q}_j$ or $\hat{Q}_A \equiv (1/n) \sum_{j \in A} \hat{q}_j$, as then the modified scaling in the correlations (G, G_{AB}, H, and H_{AB}) is exactly compensated by the modified scaling of the Heisenberg uncertainty in the numerator.

Figure 2 shows the results for $d = 0$ and $d = 1$. In the first case — if the blocks are neighboring — there exists entanglement for all possible coupling strengths α and block sizes n. In the latter case — if there is one oscillator between the blocks — due to the strongly decaying correlation functions g an h there is no entanglement between two single oscillators ($n = 1$), but entanglement for larger blocks (up to $n = 4$, depending on α). The statement that entanglement can emerge by going to larger blocks was also found in [4]. But there the blocks were abstract objects, containing all the information of their constituents. In the case of collective operators, however, increasing the block size (averaging over more oscillators) is also connected with a loss of information. In spite of this loss and the mixedness of the state, two blocks can be entangled, although non of the individual pairs between the blocks is entangled — indicating true bipartite entanglement between collective operators. For $d \geq 2$, however, no entanglement can be found anymore.

These results are in agreement with the general statement that entanglement between a region and its complement scales with the size of the boundary [15]. In

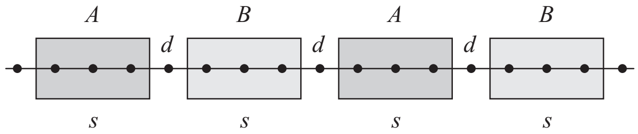

FIG. 3: Two periodic blocks of a harmonic chain A and B. Each block can consist of m subblocks with s oscillators each, separated by d oscillators. In the picture $d = 1$, $m = 2$, $s = 3$ and the number of oscillators per block is $n = ms = 6$.

the present case of two blocks in a one-dimensional chain (Fig. 1) the boundary is constant and as the blocks are made larger, the entanglement decreases since it is distributed over more and more oscillators. We therefore propose to increase the number of boundaries by considering two non-overlapping blocks, where we allow a *periodic continuation* of the situation above, i.e. a sequence of $m \geq 1$ subblocks, separated by d oscillators and each consisting of $s \geq 1$ oscillators, where $ms = n$ (Fig. 3).

The degree of entanglement between two periodic blocks of non-separated ($d = 0$) one-particle subblocks ($s = 1$) is larger for stronger coupling constant α and grows with the overall number of oscillators n (Fig. 4a). For given α and n and no separation between the subblocks ($d = 0$) the entanglement is larger for the case of small subblocks, as then there are many of them, causing a large total boundary (Fig. 4b). Entanglement can be even found for larger separation ($d = 1, 2$) with a more complicated dependence on the size s of the subblocks. There is a trade-off between having a large number of boundaries and the fact that one should have large subblocks as individual separated oscillators are not entangled (Fig. 4c,d). For $d \geq 3$ no entanglement can be found anymore. (In a realistic experimental situation, where the separation d is not sharply defined, e.g., where there are weighted contributions for $d = 0, 1, ..., d_{\max}$, entanglement can persist even for $d_{\max} \geq 3$, depending on the weighting factors.)

For the sake of completeness we give a rough approximation of the entanglement between two periodic blocks. Let us assume that the subblocks are directly neighbored, $d = 0$. Furthermore, we consider couplings α such that we may neglect higher than next neighbor correlations ($\alpha \lesssim 0.5$), i.e., we only take into account g_0, g_1, h_0 and h_1. The correlations read

$$G = \frac{1}{n} \sum_{j \in A} \sum_{i \in A} g_{|j-i|} \approx g_0 + \frac{2\,m\,(s-1)}{n}\, g_1\,, \qquad (27)$$

$$G_{AB} = \frac{1}{n} \sum_{j \in A} \sum_{i \in B} g_{|j-i|} \approx \frac{1}{n}\,(2\,m-1)\, g_1\,, \qquad (28)$$

and analogously for H and H_{AB}. The first equation reflects that there are n self-correlations and $m\,(s-1)$ nearest neighbor pairs (which are counted twice) within one block, i.e., $s-1$ pairs per subblock. The second equation represents the fact that there are $2\,m-1$ boundaries

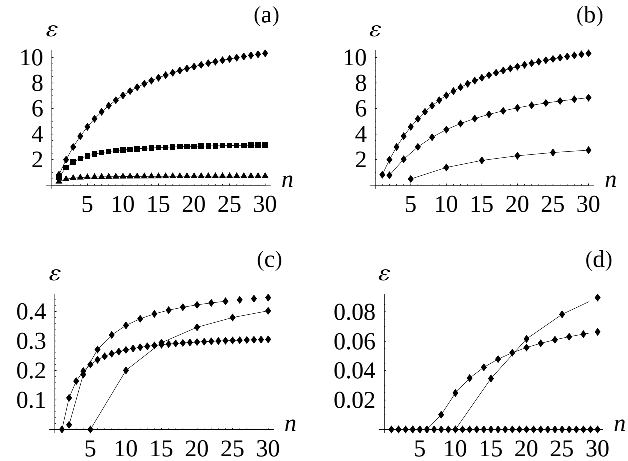

FIG. 4: Degree of collective entanglement ε for two periodic blocks of oscillators as a function of their total size n. (a) Neighboring one-particle subblocks ($d = 0$, $s = 1$). Entanglement monotonically increases with n and becomes larger as the coupling strength α increases. Plotted are $\alpha = 0.99$ (diamonds), $\alpha = 0.9$ (squares) and $\alpha = 0.5$ (triangles). (b) The coupling is fixed to $\alpha = 0.99$ for this and the subsequent figures. There is no separation, $d = 0$. Plotted are the cases $s = 1, 2, 5$. For fixed n the entanglement is more or less proportional to the number of boundaries, i.e., inversely proportional to the subblock size s. (c) and (d) correspond to the cases $d = 1$ and $d = 2$, respectively. The dependence on the size of the subblocks is more complicated as there is a trade-off between having a large number of boundaries (i.e. small s) and the fact that one should have large subblocks as individual separated oscillators are not entangled.

where blocks A and B meet. Using $s = n/m$, the entanglement (25) becomes (note that $g_1 > 0$ and $h_1 < 0$)

$$\varepsilon \approx \frac{1}{4\,[g_0 + (2 - \frac{4\,m-1}{n})\, g_1]\,[h_0 + (2 - \frac{1}{n})\, h_1]} - 1\,. \qquad (29)$$

For given n this approximation obviously increases with the total number of boundaries, m. It can be considered as an estimate for a situation like in Fig. 4b, if a smaller coupling is used such that the neglect of higher correlations becomes justified.

We close this section by annotating that the entanglement (25) between collective blocks of oscillators — being the Gaussian part — can in principle (for sufficient control of the block separation d) be transferred to two remote qubits via a Jaynes–Cummings type interaction [6, 10, 12]. For the interaction with periodic blocks "gratings" have to be employed in the experimental setup. The interaction Hamiltonian is of the form

$$\begin{aligned} \hat{H}_{\text{int}} \sim\, & (\mathrm{e}^{-\,\mathrm{i}\,\omega_1\,t}\,\hat{\sigma}_1^{+} + \mathrm{e}^{+\,\mathrm{i}\,\omega_1\,t}\,\hat{\sigma}_1^{+})\,\hat{Q}_A \\ & + (\mathrm{e}^{-\,\mathrm{i}\,\omega_2\,t}\,\hat{\sigma}_2^{+} + \mathrm{e}^{+\,\mathrm{i}\,\omega_2\,t}\,\hat{\sigma}_2^{+})\,\hat{Q}_B\,, \end{aligned} \qquad (30)$$

where ω_i is the Rabi frequency and $\hat{\sigma}_i^{+} = (\hat{\sigma}_i^{-})^\dagger = |e\rangle_i\,{}_i\langle g|$ is the bosonic operator (with $|g\rangle_i$ and $|e\rangle_i$ the ground and the excited state) of the i-th qubit ($i = 1, 2$).

V. COLLECTIVE OPERATORS FOR SCALAR QUANTUM FIELDS

The continuum limit of the linear harmonic chain is the (1+1)-dimensional Klein–Gordon field $\phi(x,t)$ with the canonical momentum field $\pi(x,t) = \dot{\phi}(x,t)$. It satisfies the Klein–Gordon equation (in natural units $\hbar = c = 1$) with mass m

$$\ddot{\phi} - \nabla^2\phi + m^2\,\phi = 0\,. \tag{31}$$

With the canonical quantization procedure ϕ and π become operators satisfying the non-trivial commutation relation $[\hat{\phi}(x,t), \hat{\pi}(x',t)] = \mathrm{i}\,\delta(x-x')$. The field operator can be expanded into a Fourier integral over elementary plane wave solutions [26]

$$\hat{\phi}(x,t) = \int \frac{\mathrm{d}k}{\sqrt{4\,\pi\,\omega_k}}\,\big[\hat{a}(k)\,\mathrm{e}^{\mathrm{i}\,k\,x-\mathrm{i}\,\omega_k\,t} + \mathrm{H.c.}\big], \tag{32}$$

$$\hat{\pi}(x,t) = -\mathrm{i}\int \frac{\mathrm{d}k\,\omega_k}{\sqrt{4\,\pi}}\,\big[\hat{a}(k)\,\mathrm{e}^{\mathrm{i}\,k\,x-\mathrm{i}\,\omega_k\,t} - \mathrm{H.c.}\big], \tag{33}$$

where k is the wave number and $\omega_k = +\sqrt{k^2+m^2}$ is the dispersion relation. The annihilation and creation operators fulfil $\big[\hat{a}(k), \hat{a}^\dagger(k')\big] = \delta(k-k')$. We write the field operator as a sum of two contributions $\hat{\phi} = \hat{\phi}^{(+)} + \hat{\phi}^{(-)}$, where $\hat{\phi}^{(+)}$ ($\hat{\phi}^{(-)}$) is the contribution with positive (negative) frequency. Thus, $\hat{\phi}^{(+)}$ corresponds to the term with the annihilation operator in (32). The vacuum correlation function is given by the (equal-time) commutator of the positive and the negative frequency part:

$$\langle 0|\,\hat{\phi}(x,t)\,\hat{\phi}(y,t)\,|0\rangle = [\hat{\phi}^{(+)}(x,t), \hat{\phi}^{(-)}(y,t)]\,. \tag{34}$$

It is a peculiarity of the idealization of quantum field theory that for $x = y$ this propagator diverges in the ground state:

$$\langle 0|\,\hat{\phi}^2(x,t)\,|0\rangle \to \infty\,. \tag{35}$$

The same is true for $\langle 0|\,\hat{\pi}^2(x,t)\,|0\rangle$ and hence we cannot easily build an entanglement measure like for the harmonic chain, since the analogs of the two-point correlation functions g_0 and h_0, (5) and (6), are divergent now. Automatically, we are motivated to study the more physical situation and consider extended space-time regions, which means that we should integrate the field (and conjugate momentum) over some spatial area. We define the collective operators

$$\hat{\Phi}_L(x_0,t) \equiv \frac{1}{\sqrt{L}}\int_{-L/2}^{L/2} \hat{\phi}(x+x_0,t)\,\mathrm{d}x\,, \tag{36}$$

$$\hat{\Pi}_L(x_0,t) \equiv \frac{1}{\sqrt{L}}\int_{-L/2}^{L/2} \hat{\pi}(x+x_0,t)\,\mathrm{d}x\,, \tag{37}$$

Therefore, $\hat{\Phi}_L(x_0,t)$ and $\hat{\Pi}_L(x_0,t)$ are equal-time operators which are spatially averaged over a length L, centered at position x_0. The commutator is

$$[\hat{\Phi}_L(x_0,t), \hat{\Pi}_L(x_0,t)] = \frac{1}{L}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \mathrm{i}\,\delta^{(3)}(x-y) = \mathrm{i}\,, \tag{38}$$

which is in complete analogy to (9). If $\hat{\Phi}_L$ and $\hat{\Pi}_L$ correspond to separated regions without overlap, i.e., $|x_0-y_0| > L$, then of course $[\hat{\Phi}_L(x_0,t), \hat{\Pi}_L(y_0,t)] = 0$. The spatial integration in (36) and (37) can be carried out analytically:

$$\hat{\Phi}_L(x_0,t) = \frac{1}{\sqrt{\pi\,L}}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k\,\sqrt{\omega_k}}\,\sin\!\left(\frac{k\,L}{2}\right) \times \big[\hat{a}(k)\,\mathrm{e}^{\mathrm{i}\,k\,x_0-\mathrm{i}\,\omega_k\,t} + \mathrm{H.c.}\big], \tag{39}$$

$$\hat{\Pi}_L(x_0,t) = \frac{-\mathrm{i}}{\sqrt{\pi\,L}}\int_{-\infty}^{\infty} \frac{\mathrm{d}k\,\sqrt{\omega_k}}{k}\,\sin\!\left(\frac{k\,L}{2}\right) \times \big[\hat{a}(k)\,\mathrm{e}^{\mathrm{i}\,k\,x_0-\mathrm{i}\,\omega_k\,t} - \mathrm{H.c.}\big]. \tag{40}$$

The final step is to calculate the propagators of the field and the conjugate momentum. We find

$$D_{\hat{\Phi},L}(r) \equiv \langle 0|\,\hat{\Phi}_L(x_0,t)\,\hat{\Phi}_L(y_0,t)\,|0\rangle \tag{41}$$
$$= \frac{1}{\pi\,L}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k^2\,\sqrt{k^2+m^2}}\,\sin^2\!\left(\frac{k\,L}{2}\right)\cos(k\,r)\,,$$

$$D_{\hat{\Pi},L}(r) \equiv \langle 0|\,\hat{\Pi}_L(x_0,t)\,\hat{\Pi}_L(y_0,t)\,|0\rangle \tag{42}$$
$$= \frac{1}{\pi\,L}\int_{-\infty}^{\infty} \frac{\mathrm{d}k\,\sqrt{k^2+m^2}}{k^2}\,\sin^2\!\left(\frac{k\,L}{2}\right)\cos(k\,r)\,,$$

with $r \equiv |x_0-y_0|$ the distance between the centers of the two regions, reflecting the spatial symmetry. Thus $D_{\hat{\Phi},L}(0)$ and $D_{\hat{\Pi},L}(0)$ are the analogs of $\langle\,\hat{Q}^2_{A,B}\,\rangle$ and $\langle\,\hat{P}^2_{A,B}\,\rangle$ (*intra*-block correlations within the same block), respectively, whereas $D_{\hat{\Phi},L}(r>L)$ and $D_{\hat{\Pi},L}(r>L)$ correspond to $\langle\,\hat{Q}_A\hat{Q}_B\,\rangle$ and $\langle\,\hat{P}_A\hat{P}_B\,\rangle$ (*inter*-block correlations between separated blocks).

The expressions (41) and (42) are finite, especially for $r = 0$. Mathematically, the integration over a finite spatial region L corresponds to a cutoff, which removes the divergence we faced in (35). However, the expressions are ill defined for $L \to 0$.

Applying the entanglement measure (25) with $G = D_{\hat{\Phi},L}(0)$, $H = D_{\hat{\Pi},L}(0)$, $G_{AB} = D_{\hat{\Phi},L}(r)$, and $H_{AB} = D_{\hat{\Pi},L}(r)$ does not indicate entanglement for any choice of L and $r > L$. The same is true for the generalized case of blocks consisting of periodic subregions of space, showing an inherent difference between the harmonic chain and its continuum limit. We believe this is due to the fact, that any spatial integration immediately corresponds to an infinitely large block in the discrete harmonic chain and that the information loss (compared to the mathematical indeed existing exponentially small entanglement [6]) due to the collective operators already is too large. Nonetheless, defining collective operators like in (36) and (37) and use of the measure (25) may reveal entanglement between spatially separated regions for other quantum field states, which is the subject of future research.

VI. CONCLUSION

Our results have importance for investigating the conditions under which entanglement can be detected by measuring collective observables of blocks consisting of a large number of harmonic oscillators. This has relevance for schemes of extracting entanglement where the probe particles normally interact with whole (periodic) groups of oscillators rather than single oscillators. The results are also relevant for the transition from the quantum to the classical domain as they suggest that entanglement between collective operators (global properties) may persist even in the limit of a large number of particles. It is obvious that our approach of collective observables can be extended to more dimensions. Furthermore, we demonstrated its potential application to scalar quantum field theory.

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